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Sequence spaces of spline functions on subsets and l^∞-spaces

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Abstract

Spaces of sequences of bounded linear splines defined on arbitrary subsets of $\mathbb{R}$ are studied, especially with respect to continuous extensions. An extension problem is solved by establishing a decomposition for the space of spline sequences with respect to the l^∞-space on a corresponding subset of $\mathbb{Z}_+ \times \mathbb{Z}$. An application to Zygmund spaces on subsets is presented.

Keywords: Zygmund space; Sequence space; Linear spline; Extension operator

1. Introduction

We discuss some Banach spaces of sequences of bounded linear spline functions on subsets, F, of $\mathbb{R}$. We are especially interested in solving the problem of constructing continuous extension operators on these spaces and we present a solution to this problem. This solution is founded on the construction of linear and bounded bijections between three spaces, denoted by $\mathscr{T}(F)$, $\mathscr{S}(F)$ and $l^\infty(D(F))$. The $\mathscr{T}(F)$-spaces are closely related to the Zygmund spaces on subsets and the $l^\infty(D(F))$-spaces are restrictions of the $l^\infty(\mathbb{Z}_+ \times \mathbb{Z})$-space to certain subsets.

The successful outcome with our method of solution depends on the bijections which transfer the difficult extension problem for $\mathscr{T}(F)$ to a trivial one for

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$l^\infty(D(F))$. We give an application of the above result, in doing so we introduce a new continuous extension operator for the Zygmund spaces on subsets of $\mathbb{R}$.

Our general idea when studying function spaces is to avoid definitions and constructions which contain uncountability. Instead, we use definitions which are based on countability and finiteness. The reasoning for this approach being that further applications can be devised with our results.

Many mathematicians have paved the way for, and contributed to, the solutions of the extension problems for the Zygmund space on subsets. Those who did the final work are Brudnyi and Shvartsman [Shv84], [Shv87] on one hand and Jonsson and Wallin on the other [JW79] [Jon85]. Their solutions are basically built on Whitney's extension operator and are modifications of it. Whitney's extension procedure is composed of several parts, one of which is to characterize the functions which one would like to extend by discontinuous functions. The latter are piecewise polynomials and the characterization is made on uncountably many intervals. We suggest that one should avoid uncountability and discontinuity by using a characterization by a sequence of smooth splines. We follow through on this idea by examining in detail the case of the Zygmund space in dimension one, and show that this case is an application of our theory for the three spaces $\mathscr{T}(F)$, $\mathscr{S}(F)$ and $l^\infty(D(F))$ and the connections between them.

The present work was preconceived in [Win01], and our result is a generalization of the main result there. In that work, we presented a decomposition for the Zygmund space on $\mathbb{R}$. The decomposition we now present is more general since it is valid for subsets F of $\mathbb{R}$. The generalization is possible thanks to a more general and improved proof.

There are a huge number of results and papers on extensions of Zygmund-, Besov- and related function spaces. As a general background for the present work we recommend works by Zygmund [Zyg89], Whitney [Whi34a] [Whi34b], Besov-Ilin-Nikolski [BIN79], Jonsson–Wallin [JW84], Brudyi–Shvartsmann [BS97], and references there. As special background, we refer to [Win01].

2. Notation, definitions and theorems

The disposition of this paragraph is as follows. We introduce our notation, define three spaces $\mathscr{T}(F)$, $\mathscr{S}(F)$ and $l^\infty(D(F))$, establish one-to-one mappings between the three spaces, extend the $l^\infty(D(F))$-space to the $l^\infty(\mathbb{Z}_+\times\mathbb{Z})$-space, and then apply the inverses and map back to $\mathscr{T}(\mathbb{R})$. That gives us an extension of $\mathscr{T}(F)$ to $\mathscr{T}(\mathbb{R})$. Finally we apply the result to the Zygmund spaces $\Lambda(1,F)$.

The following notation will be used throughout this paper:

$K_0=\mathbb{Z}$ and $K_p=2^{1-p}(\mathbb{Z}+1/2),\ p=1,2,\dots$.

$\mathscr{I}(K_0)=\{J : J=[i,i+1],\ i\,\varepsilon\,\mathbb{Z}\}$.

$\mathscr{I}(K_p)=\{J : J=2^{1-p}[i-1/2,i+1/2],\ i\,\varepsilon\,\mathbb{Z}\},\ p=1,2,\dots$.

F is a closed subset of $\mathbb{R}$.

$F(\delta)$, where $\delta>0$, is the closed δ-neighbourhood of F.

$F_p = \cup\{J : J \,\varepsilon\, \mathscr{I}(K_p) \text{ and } J(2^{1-p}) \cap F \neq \emptyset\},\ p = 0, 1, 2, \dots$.
$B(x) = dist(x, (-\infty, 0) \cup (2, \infty))$.
Following the above notation we have that

$$F(2^{1-p}) \supset F_{p+1} \supset F(2^{-p}), \quad p = -1, 0, 1, \dots$$

and hence

$$F_p \supset F_{p+1} \supset \dots \supset F, \quad p = 0, 1, 2, \dots \ . \tag{1}$$

With respect to a given F we now define three function spaces $\mathscr{U}(F)$, $\mathscr{T}(F)$ and $\mathscr{S}(F)$ where $\mathscr{U}(F) \supset \mathscr{T}(F) \supset \mathscr{S}(F)$.

Definition 1. Let $\mathscr{U}(F)$ be the space of all sequences of linear splines $\{u_p(x)\}_{p=0}^{\infty}$ where $u_0(x)$ has domain F_0 and is identically zero, $u_p(x)$ have domain F_p, knot points at $F_p \cap K_p$, $p = 1, 2, \dots$ and

$$\sup_{p=1,2,\dots,} \ \sup_{x \,\varepsilon\, F_p} |(u_p)(x)| < +\infty. \tag{2}$$

Definition 2. Let $\mathscr{T}(F) \subset \mathscr{U}(F)$ be the space of all sequences $\{\tau_p(x)\}_{p=0}^{\infty}$ such that

$$\sup_{p=0,1,2,\dots,} \ \sup_{x \,\varepsilon\, F_{p+1}} 2^p |(\tau_{p+1} - \tau_p)(x)| < +\infty. \tag{3}$$

Definition 3. Let $\mathscr{S}(F) \subset \mathscr{U}(F)$ be the space of all sequences $\{s_p(x)\}_{p=0}^{\infty}$ such that

$$\sup_{p=0,1,2,\dots,} \ \sup_{x \,\varepsilon\, F_{p+1}} 2^p |(s_{p+1})(x)| < +\infty. \tag{4}$$

Since it is obvious that $\mathscr{S}(F) \subset \mathscr{T}(F)$ and straightforward to prove the following theorem the proof is left out.

Theorem 1. *The spaces* $\mathscr{U}(F)$, $\mathscr{T}(F)$ *and* $\mathscr{S}(F)$ *are Banach spaces with norms* $||\cdot||_{\mathscr{U}(F)}$, $||\cdot||_{\mathscr{T}(F)}$ *and* $||\cdot||_{\mathscr{S}(F)}$ *given by the left-hand sides of* (2), (3) *and* (4), *respectively.*

We make the following useful definition of an operator σ_n on the space of real valued functions on F_n into the space of linear splines on F_n.

Definition 4. For a real-valued function $f(x), x \,\varepsilon\, F_n$, let $\sigma_n[f]$ be the linear spline which interpolates f and has knot points at $K_n \cap F_n$, $n = 1, 2, \dots$.

When there is no ambiguity, we refrain from writing the independent variable x in the function. For instance, we write $\{\tau_p\}_{p=0}^{\infty}$ instead of $\{\tau_p(x)\}_{p=0}^{\infty}$.

Lemma 1. *Let F be closed and* $\tau = \{\tau_p(x)\}_{p=0}^{\infty}$ *be in* $\mathcal{T}(F)$. *Then*

$$|\tau_p - \sigma_p[\tau_{p-1}]| \leqslant 2^{-(p-1)}||\tau||_{\mathcal{T}(F)} \tag{5}$$

and

$$|[I - \sigma_p][\tau_{p-1}]| \leqslant 2^{-(p-2)}||\tau||_{\mathcal{T}(F)}. \tag{6}$$

Proof. To prove (5) use Definition 2 and the three facts that $\tau_p = \sigma_p[\tau_p]$, σ_p is linear and $\sigma_p[f] = f$ on K_p. Then it follows that

$$|\tau_p - \sigma_p[\tau_{p-1}]| \tag{7}$$

$$= |\sigma_p[\tau_p - \tau_{p-1}]| \leqslant \sup_{K_p \cap F_p} |\sigma_p[\tau_p - \tau_{p-1}]| \tag{8}$$

$$= \sup_{K_p \cap F_p} |\tau_p - \tau_{p-1}| \leqslant 2^{-(p-1)}||\tau||_{\mathcal{T}(F)}. \tag{9}$$

To prove (6), we just use the triangle inequality and Definition 2. Then

$$\begin{aligned} &|[I - \sigma_p][\tau_{p-1}]| \\ &\quad = |\tau_{p-1} - \sigma_p[\tau_{p-1}]| = |\tau_{p-1} - \tau_p + \tau_p - \sigma_p[\tau_{p-1}]| \\ &\quad \leqslant |\tau_{p-1} - \tau_p| + |\sigma_p[\tau_p - \tau_{p-1}]| \leqslant 4 \times 2^{-p}||\tau||_{\mathcal{T}(F)} \end{aligned} \tag{10}$$

and the two inequalities are proved. □

Lemma 2. *Let* $t(x)$ *be a bounded linear spline with knot points in* $K_0 \cup K_1 \cup \cdots \cup K_{p-1}$ *and zeroes in* K_{p-1}. *Then*

$$\sup_{x \varepsilon \mathbb{R}} |\sigma_p[t](x)| \leqslant \tfrac{1}{2} \sup_{\mathbf{x} \varepsilon \mathbb{R}} |t(x)|. \tag{11}$$

We omit the proof since the inequality is obvious.

The following lemma and its proof are used in two of the theorems below.

Lemma 3. *Let* $s = \{s_p(x)\}_{p=0}^{\infty} \, \varepsilon \, \mathcal{S}(F)$ *and* $J \, \varepsilon \, \mathcal{I}(K_p)$ *such that* $J \, \varepsilon \, \mathcal{I}(K_p)$. *Then there is a nonnegative integer* $m < p$ *so that*

$$|[I - \sigma_p][s_0 + s_1 + \cdots + s_{p-1}](x)| = |[I - \sigma_p][s_m](x)| \leqslant 2^{1-p}||s||_{\mathcal{S}(F)}, \quad x \, \varepsilon \, J \tag{12}$$

Proof. From the construction of the knot point sets, it follows that the intervals in $\mathcal{I}(K_p)$ cover $\mathbb{R}$, have disjoint interior, and that each such interval contain exactly one point from $K_0 \cup K_1 \cup \cdots \cup K_{p-1}$ and this point is in the center of the interval. Since for a given $l < p$ the linear spline s_l has knot points exclusively in K_l, it follows that for the given interval $J \varepsilon \mathscr{I}(K_p)$ there is at most one $l < p$, say $l = m$, so that s_m has a knot point in J. All others, $s_0, s_1, \dots, s_{m-1}, s_{m+1}, \dots, s_{p-1}$ are linear in J and hence

$$[I - \sigma_p][s_l](x) = 0, \quad x \,\varepsilon\, I, \;\; l = 0, 1, \dots, m-1, m+1, \dots, p-1. \tag{13}$$

Thus the equality in (12) is proved. Now consider the inequality in (12).

From $\{s_p\}_{p=0}^{\infty} \,\varepsilon\, \mathscr{S}(F)$ and Theorem 1 we get that s_m is Lipschitz with constant $2||s||_{\mathscr{S}(F)}$ and hence

$$|\Delta_{2^{-p}} s_m| \leqslant 2||s||_{\mathscr{S}(F)} 2^{-p}. \tag{14}$$

In the estimation of $|[I - \sigma_p][s_m](x)|$ below we use z for $2^{1-p}(i + \frac{1}{2})$. Then it follows from the Lipschitz property that

$$\begin{aligned} &|[I - \sigma_p][s_m](x)|, \\ &\quad \leqslant \tfrac{1}{2} |s_m(z + 2^{-p}) + s_m(z - 2^{-p}) - 2s_m(z)| \\ &\quad \leqslant \tfrac{1}{2}(|s_m(z + 2^{-p}) - s_m(z)| + |s_m(z - 2^{-p}) - s_m(z)|) \\ &\quad \leqslant \tfrac{1}{2}(2||s||_{\mathscr{S}(F)} 2^{-p} + 2||s||_{\mathscr{S}(F)} 2^{-p}) \\ &\quad = 2^{1-p} ||s||_{\mathscr{S}(F)} \end{aligned} \tag{15}$$

and this completes the proof of the lemma. □

We now introduce a linear transformation $\Phi : \mathscr{T}(F) \to \mathscr{S}(F)$ which maps $\tau = \{\tau_i\}_0^{\infty}$ in $\mathscr{T}(F)$ to $s = \{s_i\}_0^{\infty}$ in $\mathscr{S}(F)$ according to

$$s_p = \tau_p - \sigma_p[s_0 + s_1 + \cdots + s_{p-1}], \quad p = 0, 1, 2 \dots \,. \tag{16}$$

For this transformation we have the following important theorem.

Theorem 2. *The linear transformation Φ is bijective and Φ and its inverse Φ^{-1} are bounded in the following sense.*

$$||\Phi(\{\tau_i\}_{i=0}^{\infty})||_{\mathscr{S}(F)} \leqslant 4||\{\tau_i\}_{i=0}^{\infty}||_{\mathscr{T}(F)} \tag{17}$$

and

$$||\Phi^{-1}(\{s_i\}_{i=0}^{\infty})||_{\mathscr{T}(F)} \leqslant 3||\{s_i\}_{i=0}^{\infty}||_{\mathscr{S}(F)}. \tag{18}$$

Proof. From the simple construction, given by (16), of the sequence $\{s_i\}_{i=0}^{\infty}$ by the sequence $\{\tau_i\}_{i=0}^{\infty}$ it is easy to see that Φ is linear on $\mathscr{U}(F)$, 1-1, and that the inverse is defined and linear on $\mathscr{U}(F)$. The proof will be complete when we prove (17) and (18). We prove (17) by estimating $|s_n|$, $n \,\varepsilon\, \mathbb{Z}_+$. From (16) follows

$$\begin{aligned} &s_1 + s_2 + \cdots + s_i \\ &\quad = s_1 + s_2 + \cdots + s_{i-1} + \tau_i - \sigma_i[s_1 + s_2 + \cdots + s_{i-1}] \\ &\quad = \tau_i + [I - \sigma_i][s_1 + s_2 + \cdots + s_{i-1}], \quad i = 1, 2, \dots \,. \end{aligned} \tag{19}$$

This is a recursion formula. The first step to take to estimate s_n is to express it by (16) and then rewrite it by the recursion formula. We then get

$$\begin{aligned} s_n &= \tau_n - \sigma_n[s_1 + s_2 + \cdots + s_{n-1}] \\ &= \tau_n - \sigma_n[\tau_{n-1} + [I - \sigma_{n-1}][s_1 + s_2 + \cdots + s_{n-2}]] \\ &= \tau_n - \sigma_n[\tau_{n-1} + [I - \sigma_{n-1}][\tau_{n-2} + [I - \sigma_{n-2}][s_1 + s_2 + \cdots + s_{n-3}]]]. \end{aligned} \tag{20}$$

We would like to estimate $[I - \sigma_{n-2}][s_1 + s_2 + \cdots + s_{n-3}]$ in Eq. (20). This is done by proving that

$$||[I - \sigma_i][s_0 + s_1 + s_2 + \cdots + s_{i-1}]|| \leqslant 3 \cdot 2^{-(i-1)} ||\tau||_{\mathscr{T}(F)} \tag{21}$$

is true for $i \varepsilon \mathbb{Z}_+$ by induction:

The inequality is true for $i = 1$ since $s_0 = 0$.

For $i = 2$ the left-hand side of (21) is

$$||[I - \sigma_2][s_1]|| = ||[I - \sigma_2][\tau_1]||. \tag{22}$$

Then by Lemma 1 inequality (21) is true for $i = 2$.

We now prove that (21) holds for $i = m$ assuming that it holds for $i \leqslant m - 1$. We first consider the expression $s_1 + s_2 + \cdots + s_{m-2}$ with knot points in $K_1 \cup K_2 \cup \cdots \cup K_{m-2}$ and thereafter $[I - \sigma_{m-1}][s_1 + s_2 + \cdots + s_{m-2}]$ with knot points in $K_1 \cup K_2 \cup \cdots \cup K_{m-2} \cup K_{m-1}$. We observe that the latter expression is 0 in K_{m-1} and that the set K_{m-1} coincides with the set of midpoints of those intervals which are bounded by pairs of consecutive points from $K_0 \cup K_1 \cup K_2 \cup \cdots \cup K_{m-2}$. If we then use the fact that every linear spline is uniquely determined by its sequence of functions values in the knot points then we have, from the induction assumption, that the sequence of function values at the knot points may be written

$$\ldots, a, 0, b, 0, c, 0, \ldots \tag{23}$$

and that they are bounded by the right-hand side of (21) for $i = m - 1$. We now apply the operator $[I - \sigma_m]$ to $[I - \sigma_{m-1}](s_1 + s_2 + \cdots + s_{m-2})$. After some considerations we see that this creates a new series which is

$$\ldots, \frac{a}{2}, 0, -\frac{1}{2}\left(\frac{a + b}{2}\right), 0, \frac{b}{2}, 0, \ldots \ . \tag{24}$$

We then add (compare (20))

$$[I - \sigma_m][\tau_{m-1}] \tag{25}$$

to

$$[I - \sigma_m][I - \sigma_{m-1}][s_0 + s_1 + \cdots + s_{m-2}]. \tag{26}$$

Expression (25) corresponds to a sequence

$$\ldots, \alpha, 0, 0, \beta, 0, 0, \gamma, 0, 0, \ldots, \tag{27}$$

where $\ldots\alpha, \beta, \gamma\ldots$ appear where sequence (24) has values of type $-\frac{1}{2}(\frac{a+b}{2})$ and they are bounded according to Lemma 1, that is by $2^{-m} \cdot 4 \cdot ||\tau||_{\mathscr{T}(F)}$. Hence we get

$$[I - \sigma_m][\tau_{m-1}] - [I - \sigma_m][I - \sigma_{m-1}][s_0 + s_1 + \cdots + s_{m-2}] \tag{28}$$

and the corresponding sequence

$$\ldots, \frac{a}{2}, 0, -\frac{1}{2}\left(\frac{a+b}{2}\right) + \alpha, 0, \frac{b}{2}, 0, \ldots \ . \tag{29}$$

From Lemma 1 in [Win01] and from the induction assumption above it then follows that sequence (29) is bounded by $3 \cdot 2^{-(m-1)}||\tau||_{\mathscr{T}(F)}$. By induction it now follows that (21) is true for all natural numbers, in particular for $i = n - 1$. To conclude the proof of (17), we estimate, via estimations of the right-hand side of (20); that is, by estimating

$$(\tau_n - \sigma_n[\tau_{n-1}]) - (\sigma_n[I - \sigma_{n-1}]\ldots). \tag{30}$$

By Lemma 1, the first term is bounded as

$$|\tau_n - \sigma_n[\tau_{n-1}]| \leqslant 2^{-(n-1)}||\tau||_{\mathscr{T}(F)}. \tag{31}$$

and by Lemma 2 and (21), the second term is bounded as

$$|\sigma_n[I - \sigma_{n-1}]\cdots| \leqslant \frac{1}{2} \cdot 3 \cdot 2^{-(n-2)}||\tau||_{\mathscr{T}(F)}. \tag{32}$$

Hence (17) is true.

We now prove (18). That is, given $s = \{s_i\}_{i=0}^{\infty}$ with norm $||s||_{\mathscr{S}(F)}$, we prove that $\tau = \{\tau_i\}_{i=0}^{\infty} = \Phi^{-1}(\{s_i\}_{i=0}^{\infty})$ satisfies

$$||\{\tau_i\}_{i=0}^{\infty}||_{\mathscr{T}(F)} \leqslant 2||s||_{\mathscr{S}(F)}. \tag{33}$$

Definition 2 leads us to estimate $2^p|[\tau_{p+1} - \tau_p](x)| < +\infty$.

Then

$$\begin{aligned} &2^p|(\tau_{p+1} - \tau_p)(x)| \\ &\quad = 2^p|s_{p+1} + \sigma_{p+1}[s_0 + s_1 + \cdots + s_p] \\ &\qquad - (s_p + \sigma_p[s_0 + s_1 + \cdots + s_{p-1}])| \\ &\quad = 2^p|s_{p+1} - [I - \sigma_{p+1}][s_p] + [\sigma_{p+1} - \sigma_p][s_0 + s_1 + \cdots + s_{p-1}]|. \end{aligned} \tag{34}$$

We now estimate the three terms between the absolute value signs. We first get by the Definition 2 that

$$2^p|s_{p+1}(x)| \leqslant ||s||_{\mathscr{S}(F)}. \tag{35}$$

Then we study $[I - \sigma_{p+1}][s_p]$ on each $\mathscr{I}(K_p)$ interval and observe that on every second such interval $[I - \sigma_{p+1}](s_p) \equiv 0$, and on every other interval $[I - \sigma_{p+1}](s_p)$ is a hat function having zeroes at the end points and extreme values at the center. The absolute value of the extreme value is then bounded by $2^{-p}||s||_{\mathscr{S}(F)}$.

We now consider the third term $2^p|[\sigma_{p+1}-\sigma_p][s_0+s_1+\cdots+s_{p-1}]|$. Due to the distribution of knot points, we have that the $\mathscr{I}(K_p)$ intervals $2^{1-p}[k-(\frac{1}{2}),k+(\frac{1}{2})]$, $k\,\varepsilon\,\mathbf{Z}$ cover the real line and that an arbitrary such interval, say I, contains at most one knot point which belongs to one of $s_0, s_1, \dots, s_{p-1}$, say s_l, $l\leqslant p-1$. For all other indices k, $k\neq l$, we have $[\sigma_{p+1}-\sigma_p][s_l]\equiv 0$. Then we just have to estimate $|[\sigma_{p+1}-\sigma_p][s_l]|$ on I. This function is zero at the endpoints of I, it is increasing on the first fourth of I. Thereafter it is nonincreasing and its increasing rate is at most $2\cdot||s||_{S(F)}$. Since I has length 2^{1-p} we get the following bound on I.

$$|[\sigma_{p+1}-\sigma_p][s_l]|\leqslant\tfrac{1}{4}\cdot 2^{1-p}\cdot 2\cdot||s||_{\mathscr{S}(F)}=2^{-p}||s||_{\mathscr{S}(F)}. \tag{36}$$

Summing up gives

$$2^p|[\tau_{p+1}-\tau_p][s_l]|\leqslant 3||s||_{\mathscr{S}(F)}. \tag{37}$$

Then (18) is proved and therewith the proof of the Theorem is complete. □

For F we define a subset $D(F)$ of $\mathbb{Z}_+\times\mathbb{Z}$ and a corresponding l^∞-space which we denote $l^\infty(D(F))$. We prove that $\mathscr{S}(F)$ and $l^\infty(D(F))$ are isometric. The transformation which establishes the isometry is denoted Ψ. We now perform the above in detail.

We recall that for each positive integer p, the knot points in K_p are created by the formula $2^{1-p}(i+\frac{1}{2})$, $i\varepsilon\mathbb{Z}$. We introduce the notations $int(F_p)$ and $D(F)$ for the corresponding integers, i.e.

$$int(F_p):=\left\{i: 2^{1-p}\left(i+\frac{1}{2}\right)\varepsilon F_p\cap K_p\right\}$$

and

$$D(F):=\bigcup_{p=0}^{\infty}\{(p,i)\colon\ i\,\varepsilon\, int(F_p)\},$$

respectively. For each sequence $s=\{s_p(x)\}_{p=0}^{\infty}$ in $\mathscr{S}(F)$ there is a unique sequence

$$\Psi(s)=\{\{c_{(p,i)}\}_{i\,\varepsilon\, int(F_p)}\}_{p=1}^{\infty} \tag{38}$$

where

$$s_p(x)=\sum_{i\,\varepsilon\, int(F_p)} c_{p,i}2^{1-p}B(2^{p-1}x-i). \tag{39}$$

Hence Ψ is a linear transformation from $\mathscr{S}(F)$ to $l^\infty(D(F))$. Since for a given positive integer p, $\{B(2^{1-p}x-i)\}_{i=-\infty}^{\infty}$ is a basis for the space of linear splines with knot points at K_p we have that Ψ is a $1-1$ linear transformation on $\mathscr{S}(F)$ onto $l^\infty(D(F))$. Since

$$||\{\{c_{(p,i)}\}_{i\,\varepsilon\, int(F_p)}\}_{p=1}^{\infty}||_\infty=\sup_{(p,i)\,\varepsilon\, D(F)}|c_{p,i}|$$

and

$$\sup_{(p,i)\,\varepsilon\, D(F)} |c_{p,i}| = \sup_{p=1,2,\dots,}\ \sup_{x\,\varepsilon\, F_p} 2^p |s_{p+1}(x)| = ||s||_{\mathscr{S}(F)}$$

then Ψ is an isometry. Thus we have the following theorem.

Theorem 3. *The linear transformation*

$$\Psi : \mathscr{S}(F) \to l^\infty(D(F)) \tag{40}$$

is an isometry.

Now define an extension operator on the $l^\infty(D(F))$-spaces.

Definition 5. Let

$$E\colon l^\infty(D(F)) \to l^\infty((\mathbb{Z}_+ \cup \{0\}) \times \mathbb{Z})$$

where

$$E(\{\{c_{(p,i)}\}_{i\,\varepsilon\, int(F_p)}\}_{p=1}^\infty) = \{\{e_{(p,i)}\}_{i\,\varepsilon\, int(\mathbb{R}_p)}\}_{p=0}^\infty$$

and

$$e_{p,i} = \begin{cases} c_{p,i} & \text{if } i\,\varepsilon\, int(F_p), \\ 0 & \text{if } i\,\varepsilon\, \mathbb{Z}\backslash int(F_p). \end{cases}$$

It is obvious that E is linear and that its norm is 1.

Theorem 4. *The transformation*

$$E : l^\infty(D(F)) \to l^\infty((\mathbf{Z}_+ \cup \{0\}) \times \mathbf{Z})$$

is a linear extension operator and its norm $||E|| = 1$.

We may now combine earlier results to get the following theorem.

Theorem 5. *The transformations* Φ, Ψ, *and E and their inverses may be composed to an extension operator*

$$\mathscr{E} = \Phi^{-1} \circ \Psi^{-1} \circ E \circ \Psi \circ \Phi \tag{41}$$

on $\mathscr{T}(F)$ *to* $\mathscr{T}(\mathbb{R})$. *The extension is linear, bounded and* $||\mathscr{E}|| \leqslant 12$.

Proof. It follows from the definitions of Φ, Ψ, E, and the inverses Φ^{-1} and Ψ^{-1} that they are composable. Since they are linear and bounded, the composition is linear and bounded. From the bounds of the norms given in Theorems 2 and 4 it follows

$$\begin{aligned} ||\mathscr{E}(\tau)||_{\mathscr{T}(R)} &= ||\Phi^{-1}\circ\Psi^{-1}\circ E\circ\Psi\circ\Phi(\tau)|| \\ &\leqslant ||\Phi^{-1}||||\Psi^{-1}||||E||||\Psi||||\Phi||||\tau||_{\mathscr{T}(F)} \leqslant 12||\tau||_{\mathscr{T}(F)}. \end{aligned} \quad (42)$$

Thus the theorem is proved. □

We now give an application. We begin by introducing an equivalence relation.

Definition 6. The sequences τ_1 and τ_2 in $\mathscr{T}(F)$ are equivalent if they have the same limit. Hence for each limit there is an equivalance class and the set of equivalance classes is denoted by $\mathscr{T}(F)'$. The space of the limit functions is denoted by $Zyg(F)$.

We define a norm on $Zyg(F)$ by

$$||f|| := \inf\{||\tau||_{\mathscr{T}(F)} : \tau = \{\tau_i\}_{i=0}^{\infty} \text{ and } \lim \tau_i = f\}. \quad (43)$$

The existence of the bounded extension operator in Theorem 5 induces in a natural way one for $Zyg(F)$ to $Zyg(\mathbb{R})$. Now we are ready to state the relation between $\Lambda(1,F)$ and $Zyg(F)$.

Theorem 6. $\Lambda(1,\mathbb{R}) = Zyg(\mathbb{R})$. *The spaces have equivalent norms.*

Proof. Assume first that $f(x)\,\varepsilon\,\Lambda(1,\mathbb{R})$. Then from the definition of $\Lambda(1,\mathbb{R})$ (see for instance [Win01]) we know that

$$|f(x)| \leqslant ||f||_{\Lambda(1,\mathbb{R})} \quad (44)$$

and

$$|\Delta_h^2 f(x)| \leqslant ||f||_{\Lambda(1,\mathbb{R})} \cdot h \quad (45)$$

for $x\varepsilon\mathbb{R}$ $h\varepsilon(0,1]$. Then let $\tau = \{\tau_i(x)\}_{i=0}^{\infty}$ be given by $\tau_0(x) = 0$ and $\tau_n(x) = \sigma_n[f](x)$, $x\varepsilon\mathbb{R}$, $n = 1,2,\ldots$. It follows that τ converge to f. Notice that $|\sigma_{n+1}[f] - \sigma_n[f]|$ is a linear spline and hence has its maximum values at the knot points in K_n and K_{n+1}. It follows that we just have to estimate the values at one arbitrary knot point in K_n and one in K_{n+1}. Let q be an arbitrary knot point at K_n. Then

$$\sigma_n[f](q) = f(q) \quad (46)$$

and $\sigma_{n+1}[f]$ is linear in the $\mathscr{I}(K_{n+1})$-interval where q is center point. Hence $\sigma_{n+1}[f](q)$ is the arithmetic mean of $f(x)$ at the end points of that interval. We have then for $h = 2^{-(n+1)}$ that

$$\begin{aligned} |\sigma_{n+1}[f](q) - \sigma_n[f](q)| &\leqslant \sup_{x,h} \tfrac{1}{2}|\Delta_h^2 f(x)| \\ &\leqslant \tfrac{1}{2}\cdot 2^{-(n+1)}||f||_{\Lambda(1,\mathbb{R})} = \tfrac{1}{4}\cdot 2^{-n}||f||_{\Lambda(1,\mathbb{R})}. \end{aligned} \quad (47)$$

By the same type of argument, using two steps at an arbitrary knot point r in K_{n+1} we get

$$|\sigma_{n+1}[f](r) - \sigma_n[f](r)| \leqslant 2^{-(n+1)}||f||_{\Lambda(1,\mathbb{R})} + \tfrac{1}{2}\cdot 2^{-(n+1)}||f||_{\Lambda(1,\mathbb{R})}$$
$$= \tfrac{3}{4}\cdot 2^{-n}||f||_{\Lambda(1,\mathbb{R})}. \tag{48}$$

Hence

$$|\sigma_{n+1}[f](x) - \sigma_n[f](x)| \leqslant \tfrac{3}{4}2^{-n}||f||_{\Lambda(1,\mathbb{R})} \tag{49}$$

and

$$||f(x)||_{Zyg(\mathbb{R})} \leqslant \tfrac{3}{4}||f||_{\Lambda(1,\mathbb{R})}. \tag{50}$$

We now proceed to the estimation which constitutes the second half of the proof. Assume that $g(x)\,\varepsilon\,Zyg(\mathbb{R})$ with norm $||g||_{Zyg(\mathbb{R})}$. Hence for $\varepsilon > 0$ there is a sequence $\tau = \{\tau_i\}_{i=0}^{\infty}\,\varepsilon\,\mathscr{T}(\mathbb{R})$, with $\lim_{i\to\infty}\tau_i(x) = g(x)$, such that

$$||\tau||_{\mathscr{T}(\mathbb{R})} < ||g||_{Zyg(\mathbb{R})} + \varepsilon. \tag{51}$$

Then

$$2^n|\tau_{n+1} - \tau_n| < ||g||_{Zyg(\mathbb{R})} + \varepsilon. \tag{52}$$

This gives

$$2^n|\tau_n - g| \leqslant 2(||g||_{Zyg(\mathbb{R})} + \varepsilon) \tag{53}$$

and for $n = 0$

$$|g(x)| \leqslant 2(||g||_{Zyg(\mathbb{R})} + \varepsilon). \tag{54}$$

Now, let $x\,\varepsilon\,\mathbb{R}$ and $h\,\varepsilon\,(0, \frac{1}{8}]$. Then there is an integer $n \geqslant 3$ such that

$$2^{-n} < 2h \leqslant 2^{1-n} \tag{55}$$

and the interval $I_0 = [x, x+2h]$ is a subset of an $\mathscr{I}(K_{n-1})$- or an $\mathscr{I}(K_{n-2})$-interval. Hence

$$\Delta_h^2\tau_i(x) = 0 \quad \text{for } i = n-1 \text{ or } i = n-2. \tag{56}$$

Estimating in the less advantageous case ($i = n-2$) and using (53) and (54), we get

$$|\Delta_h^2 g(x)| = |\Delta_h^2[g(x) - \tau_{n-2}(x)]| = 2\cdot|\tfrac{1}{2}\Delta_h^2[g(x) - \tau_{n-2}(x)]|$$
$$\leqslant 2\cdot 2^{-(n-2)}2(||g||_{Zyg(\mathbb{R})} + \varepsilon) \leqslant 32(||g||_{Zyg(\mathbb{R})} + \varepsilon)h. \tag{57}$$

The estimation in the almost trivial case $h \geqslant \frac{1}{8}$ is done by means of (54),

$$|\Delta_h^2 g| = 2\cdot|\tfrac{1}{2}\Delta_h^2 g| \leqslant 2\cdot 4\cdot(||g||_{Zyg(\mathbb{R})} + \varepsilon) \leqslant 64h(||g||_{Zyg(\mathbb{R})} + \varepsilon). \tag{58}$$

Hence $g(x)\varepsilon\Lambda(1,\mathbb{R})$ and

$$||g(x)||_{\Lambda(1,\mathbb{R})} \leqslant 64(||g||_{Zyg(\mathbb{R})} + \varepsilon). \tag{59}$$

By that the proof is complete. □

We have the following decomposition and reconstruction of $f\,\varepsilon\,Zyg(F)$.

Theorem 7. *Let $f \varepsilon Zyg(F)$ and $\epsilon > 0$. Then there is a sequence $\{c_{p,i} : (p,i) \varepsilon D(F)\}$ so that*

$$f(x) = \sum_{(p,i)\varepsilon D(F))} c_{p,i} 2^{1-p} B(2^{p-1}x - i) \tag{60}$$

and

$$||\{c_{p,i} : (p,i)\varepsilon D(F)\}||_{l^\infty(D(F))} \leqslant 4(||f||_{Zyg(F)} + \epsilon). \tag{61}$$

Proof. Let $f \,\varepsilon\, Zyg(F)$ and $\epsilon > 0$. Then by Definition 6 for $\epsilon > 0$ there is a sequence $\tau = \{\tau_p\}_{p=0}^{\infty} \,\varepsilon\, \mathscr{T}(F)$ such that

$$\lim_{p \to \infty} \tau_p(x) = f(x), \quad x \,\varepsilon\, F. \tag{62}$$

and

$$||\tau||_{\mathscr{T}(F)} < (||f||_{Zyg(F)} + \epsilon). \tag{63}$$

Let $s = \Phi(\tau)$. From Lemma 3 and Theorem 2 follow that

$$\begin{aligned} &|s_1 + s_2 + \cdots + s_{p-1} - \sigma_p(s_1 + s_2 + \cdots + s_{p-1})| \\ &\quad \leqslant 2^{1-p}||s||_{\mathscr{S}(F)} \leqslant 8 \cdot 2^{-p}||\tau||_{\mathscr{T}(F)} \end{aligned} \tag{64}$$

and by (16) we get

$$|s_1 + s_2 + \cdots + s_{p-1} + s_p - \tau_p| \leqslant 8 \cdot 2^{-p}||\tau||_{\mathscr{T}(F)} \tag{65}$$

and

$$f(x) = \sum_{p=1}^{\infty} s_p(x), \quad x \,\varepsilon\, F. \tag{66}$$

Now, for each $s_p(x)$ there is a unique decomposition (compare (38) and (39)) and hence

$$f(x) = \sum_{(p,i) \,\varepsilon\, D(F))} c_{p,i} 2^{1-p} B(2^{p-1}x - i). \tag{67}$$

Thus (60) is true and

$$\{c_{p,i} : (p,i) \,\varepsilon\, D(F)\} = \Psi \circ \Phi(\tau), \tag{68}$$

where $||\Phi|| \leqslant 1$ and $||\Psi|| \leqslant 4$. Thus (61) is true and the proof is complete. □

Theorem 8. *$\Lambda(1,F) = Zyg(F)$ and the spaces have equivalent norms.*

Proof. From Theorem 6 we have that $\Lambda(1,\mathbb{R}) = Zyg(\mathbb{R})$. If we restrict to F and use the well known trace result $\Lambda(1,F) = \Lambda(1,\mathbb{R})|_F$ (see, e.g. [JW84]) it remains to prove that $Zyg(F) = Zyg(R)|_F$.

It is obvious from Definition 6 that

$$||f|_F||_{Zyg(F)} \leqslant ||f||_{Zyg(\mathbb{R})}. \tag{69}$$

Assume now that $f \varepsilon Zyg(F)$. Then given $\varepsilon > 0$ there is, by the definition of $Zyg(F)$ and its norm, a sequence $\tau \varepsilon \mathscr{T}(F)$ such that

$$||\tau||_{\mathscr{T}(F)} \leqslant (1+\epsilon)||f||_{Zyg(F)}. \tag{70}$$

By Theorem 5 we have

$$||\mathscr{E}(\tau)||_{\mathscr{T}(R)} \leqslant 12||\tau||_{\mathscr{T}(F)} \leqslant 12(1+\epsilon)||f||_{Zyg(F)}. \tag{71}$$

Since $\mathscr{E}(\tau)$ is the extension of $\tau \varepsilon \mathscr{T}(F)$ to $\mathscr{T}(R)$ and since f by Theorem 7 and its proof is equal to the limit of the convergent sequence τ then the natural choice for the extension $e(f)$ of f to $Zyg(R)$ is the limit of the convergent sequence $\mathscr{E}(\tau)$. Hence it follows that $e(f)|_F$ is the identity mapping on $Zyg(F)$ and the proof is complete. □

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